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Riemann–Cartan–Weyl Quantum Geometry, I: Laplacians and Supersymmetric Systems

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In this first article of a series dealing with the geometry of quantum mechanics, we introduce the Riemann–Cartan–Weyl (RCW) geometries of quantum mechanics for spin-0 systems as well as for systems of nonzero spin. The central structure is given by a family of Laplacian (or D'Alembertian) operators on forms of arbitrary degree associated to the RCW geometries. We show that they are conformally equivalent with the Laplacian operators introduced by Witten in topological quantum field theories. We show that the Laplacian RCW operators yield a supersymmetric system, in the sense of Witten, and study the relation between the RCW geometries and the symplectic structure of loop space. The RCW family of Laplacians are the infinitesimal generators of diffusion processes on nondegenerate space-times of systems of arbitrary spin.

INTRODUCTION

There are several problems in physics which cannot be considered as settled definitely. One of them is the determination of the geometrical structures to account for gravitation and nonlinear gauge theories. It has become known recently that the solutions of the monopole equations have a remarkable dependence on the sign of the scalar curvature of the metric of the four-dimensional manifold (Witten, 1994). This might appear as rather strange, given that the nonlinear *non-Abelian gauge* theories were conceived to account for "internal" degrees of freedom.

The theory of gravitation admits extensions to Cartan geometries with torsion, so that the metric structure appears as partially describing gravitation (Cartan and Einstein, 1979); moreover, the Cartan geometries appeared in

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the context of the Poincaré group theory of the theory of gravitation (Sciama, 1962; Hehl *et al.*, 1976, 1995; Changgui and Dehnen, 1991).

Another problem is the formulation of quantum mechanics in terms of trajectories of Brownian processes and the establishment of correlations described by the quantum potential (Bohm, 1952; Bohm and Vigier, 1953; Holland, 1993).

It is remarkable that both problems are connected through the Cartan geometries, specifically those which have a torsion tensor which reduces to its trace Q further described as a trivial Weyl 1-form: $Q = d \ln \psi$, with ψ a positive scalar field defined on space-time. We call these geometries RCW (for Riemann–Cartan–Weyl) geometries. We showed in Rapoport (1991) that the Weyl trace-torsion 1-form accounts for the average displacement of the most general diffusion process on space-time, while the metric describes the covariance of these processes. Both together, i.e., the first and second moments of the probability density of the processes, determine all higher moments. Yet, what is peculiar in this description is that there exists a single geometrical Laplacian operator which incorporates these two moments, and thus determines *all* the statistical profiles of the diffusions. This operator, which is the infinitesimal generator of the diffusions, and is the most general invariant second-order elliptic operator on a smooth manifold (when one assumes conservation of probability), is the Laplacian operator associated to the RCW geometry (Rapoport, 1991, 1995b, c). This operator does not only determine the probability density of the diffusions, yet its path integral representation through the Onsager–Machlup Lagrangian, and ultimately the classical smooth approximations which with maximal probability realize the diffusions (Rapoport, 1995a,b). One finds that these realizations are deviations of the geodesic flow due to the Weyl torsion, yet this does not conflict with the principle of equivalence, since the diffusions represent an interacting ensemble (Rapoport, 1995b).

In this article, we introduce the RCW geometries and their associated Laplacian operators, not just on spin-0 systems, still defined on differential forms of arbitrary degree. It was observed by Witten that geometrical Laplacians on forms of arbitrary degree are basic examples of supersymmetric systems, in which forms of odd (even) degree are fermions (bosons). We shall see that the RCW Laplacians on forms are conformally conjugate to Witten's deformed Laplacian in topological quantum field theories (TQFT) (Witten, 1982). This is quite striking, since in TQFT the field ψ is rather a functional on the infinite-dimensional loop space, not just the solution of the conformal invariant equation on space-time, as turns out to be the case in the present theory (Rapoport, 1995a, b, c). Yet, we shall see that the role of the RCW geometry is essential to the definition of the general symplectic structure on loop space, which, as is well known in the Riemannian case, is the key to obtaining exact representations of the trace of the Riemannian heat kernels, and ultimately leads to a direct proof of the Atiyah–Singer index theorem (Atiyah, 1985).

We close this introduction with the description of the origin of the method which gives Cartan's geometry the universality to describe classical spinning test-particle systems (Rapoport and Sternberg, 1984a, b) as well as interacting quantum ensembles undergoing diffusions generated by the RCW Laplacian.

In Cartan's approach to classical mechanics, as explained in Rapoport and Sternberg (1984a, b), we start with a little group H embedded as a closed subgroup in a bigger group G such that the homogeneous space G/H has the same dimension as the configuration manifold M. Here, H and G are, respectively, the Lorentz and Poincaré groups, and M is space-time, or we can take the de Sitter group $O(1, 4)$ instead of the Poincaré group and then M is a de Sitter space-time. It is the soldering form which infinitesimally identifies both spaces. Of course, Cartan was thinking of dynamical systems with classical (i.e., smooth) trajectories. The whole point of our theory as seen from Cartan's concept is its applicability to quantum fluctuations with continuous but nondifferentiable Brownian paths. Instead of copying classical dynamical systems on homogeneous space to space-time, the quantum counterpart of Cartan's method is the copying through an RCW connection on space-time of a *standard* Wiener process. Yet, to deal with Brownian paths, Cartan's calculus on manifolds is obviously not applicable and instead one has to apply the Itô stochastic calculus, or the Stratonovich calculus, which obeys the same rules for derivatives as the classical calculus in R^n, thus establishing a stochastic calculus which gives the stochastic extension of Cartan's method. The relation between classical and quantum motions is established, as described above, by the Onsager–Machlup Lagrangian representation of the transition density of the RCW diffusion. The classical system appears thus as the most probable approximation of quantum diffusion. This establishes the universality of Cartan's method in describing also the relation between quantum and classical motions.

This article is organized in the following way. We first introduce the Cartan soldering form and the Riemann–Cartan (RC) geometries, and particularly the RCW geometries, which we introduce from the point of view of conformal transformations. We then introduce the Laplacian operators associated to the RC geometries, and see that to obtain a one-to-one correspondence (in general dimension other than 2) between geometries and Laplacians we need to restrict the theory to RCW geometries. We then generalize the Laplacian to differential forms, through the introduction of Dirichlet quadratic forms to differential forms of arbitrary degree, to further introduce Witten's deformed Laplacian. Later we see that the RCW Laplacians yield a supersymmetric system which is conformally conjugate to Witten's Laplacian system. We finally discuss the symplectic structure of loop space and its relation with the RCW geometries. Finally, as preparation for a forthcoming article on the stochastic extension of Cartan's classical method, we give a local description of Cartan's classical method.

1. CARTAN CONNECTIONS AND RIEMANN–CARTAN–WEYL STRUCTURES

We shall follow here the presentation due to Rapoport (1991) and Rapoport and Sternberg (1994a,b).

We recall some basic facts and definitions in order to establish notation. Let G be a Lie group and M a differentiable manifold. A principal G-bundle over M is a manifold P on which G acts freely on the right, and such that the quotient of this G action is M. Thus we have a smooth map $\pi: P \to M$ and $\pi^{-1}(x)$ is a G orbit for each $x \in M$. We also assume that P is locally trivial in that about each x there is a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ (with the obvious definition of isomorphism). We let R_a: $P \to P$ denote right multiplication by a^{-1},

$$R_a(p) = pa^{-1}, \qquad p \in P, \ \ a \in G$$

so that R_a gives a left action of G on P,

$$R_{ab} = R_a R_b$$

Let F be some differentiable manifold on which G acts on the left. We can then form the quotient of the product space $P \times F$ by the G action; call it $F(P)$. Let π_F: $F(P) \to M$ denote the bundle projection, and ρ: $P \times F \to F(P)$ denote the passage to the quotient. Then $F(P)$ is also fibered over M by

$$\pi_F(\rho(p, f)) = \pi(p)$$

$F(P)$ is called the associated bundle (to the G action on F and the principal bundle P).

Let f: $P \to F$ be a smooth function satisfying

$$f(pa) = a^{-1}f(p) \tag{1.1}$$

Then

$$a(p, f(p)) = (pa^{-1}, af(p)) = (pa^{-1}, f(pa^{-1}))$$

Hence $\rho(p, f(p))$ is independent of the choice of $p \in \pi^{-1}(x)$. In other words, defines a section s of $F(P)$, i.e., a map

$$s\colon\quad M \to F(P) \qquad \pi \circ s = \mathrm{id}$$

$$s(x) = \rho(p, f(p)) \qquad \pi(p) = x$$

Conversely, given a section s, we may define the function f by the preceding equations and f satisfies $f(pa) = a^{-1}f(p)$. Thus:

Lemma 1.1. We have an identification of the space of sections of $F(P)$ with the space of maps $f\colon P \to F$ satisfying $f(pa) = a^{-1}f(p)$.

There are two especially important cases of this construction.

In the first case, suppose that $F = G/H$, where H is a closed subgroup of G. So $F(P)$ is a bundle of homogeneous spaces. Let $f\colon P \to F$ satisfy the identity (1.1) and thus be equivalent to a section s of $F(P)$. Consider

$$f^{-1}(H) = \{p \in P/f(p) = H \in G/H\}$$

If $p \in f^{-1}(H)$, then $f(pa) = a^{-1}H = H$ if and only if $a \in H$. Thus

$$f^{-1}(H) = P_H$$

is an H subbundle of P, a reduction of the principal G bundle to an H bundle. Conversely, suppose that P_H is an H subbundle of P. Then define $f\colon P \to G/H$ by $f(P_H) = H$ and if

$$p = qa, \qquad q \in P_H$$

then

$$f(p) = a^{-1}H$$

This is well defined, as can easily be checked, and defines a function f satisfying the condition (1.1). Thus:

A section of the bundle $(G/H)(P)$

is the same as a reduction of P to an H bundle P_H (1.2)

A second important case is where F is a vector space and the action of G is linear. Then $F(P)$ is a vector bundle. In this case we can consider k-forms Ω on P with values in F. We can consider forms which are horizontal in the sense that

$$i(\xi)\Omega = 0 \qquad \text{for any vertical tangent vector } \xi \tag{1.3}$$

where *vertical* means tangent to the fiber, and $i(\xi)$ denotes the operator of inner product with the vector field ξ. We can also consider forms which are equivariant in the sense that (here R_a^* stands for the adjoint of the tangent extension of R_a)

$$R_a^*\Omega \equiv \Omega \circ dR_a = a\Omega \qquad \text{for all} \quad a \in G \tag{1.4}$$

It is easy to check that (1.1) generalizes to:

Lemma 1.2. An f-valued k-form on P satisfying (1.3) and (1.4) is equivalent to an $F(P)$-valued k-form on M.

[The case $k = 0$ of Lemma 1.2 is then (1.1)].

We can combine the preceding two cases. Suppose that we are given a section s of $(G/H)(P)$, so we get a reduced bundle, P_H. We can consider the vector bundle associated to the adjoint action of H on g/h, the homogeneous space given by the Lie algebras g and h of G and H, respectively. This vector bundle can be identified with the bundle of vertical tangent vectors to $(G/H)(P)$ along the section s. Indeed, we can consider $(G/H)(P)$ as the bundle associated to P_H relative to the H action on G/H. On the principal bundle P_H the section s corresponds to the identically constant function $f \equiv H$. At the point H we have an identification of $T(G/H)$ with g/h. This identification is consistent with the H action. Thus we may identify (g/h)(P_H) with the bundle of vertical tangent vectors to $(G/H)(P)$ along s. Now suppose that Θ: $TP \to$ g/h is a 1-form which satisfies (1.3) and (1.4) relative to the group H. Then Θ can be thought of as a 1-form on M with values in (g/h)(P_H). Thus, let s be a section of $(G/H)(P)$ and P_H the corresponding reduced bundle. Let Θ be a 1-form on P_H with values in g/h which satisfies (1.3) and (1.4) relative to the group H. Then:

Θ can be regarded as a 1-form on M
with values in the bundle of vertical tangent
vectors to $(G/H)(P)$ along s (1.5)

In the particular case that dim G/H = dim M, we can further demand that the 1-form Θ on M give an isomorphism between TM and the bundle of vertical tangent vectors. This is the method conceived by Cartan. The form Θ is called a *soldering form* (or "soudage" in French).

For example, suppose that $H = O(V)$ is the orthogonal group of a vector space with a nondegenerate scalar product and $G = H \oplus V$ (semi-direct sum). Then $G/H = V$. A soldering form Θ then gives an identification of TM with $V(P_H)$. In particular, this puts a (pseudo or) Riemannian metric on M and also allows us to identify P_H with the bundle of orthogonal frames. Similarly, if we take $H = Gl(V)$, then a soldering form Θ allows us to identify P_H with the bundle of all linear frames on M.

Conversely, let P_H denote the bundle of frames of a differentiable manifold M, where $H = Gl(R^n)$, n = dim M. Then P_H carries a canonical 1-form Θ valued in R^n, and Θ satisfies (1.2) and (1.3). Namely, Θ is the so-called "structure form" defined by

$$\Theta(\xi) = p(d\pi_p \xi), \qquad \xi \in TP_H$$
$$p\colon \quad T_{\pi(p)}M \to R^n, \qquad p \in P_H \tag{1.6}$$

Of course, we can enlarge the bundle P_H to a G-bundle $P_G = G(P_H)$ and then we are back at the situation described above.

Let P be a vector bundle with structure group G. For each $\xi \in$ g, the Lie algebra of G, let ξ denote the corresponding vector field on P_G given by the right action of G on P. Recall that a connection on P_G can be described as a g-valued 1-form ω_G on P_G which satisfies

$$i(\hat{\xi})\omega = \xi, \qquad \xi \in g \tag{1.7}$$

and

$$R_a^*\omega = Ad_a\omega_G \tag{1.8}$$

The *horizontal space* of a connection ω_G at a point $p \in P_G$ consists of all tangent vectors at p which are annihilated by ω_G, i.e., those which satisfy

$$i(\nu)\omega_G = 0 \tag{1.9}$$

Any curve on M lifts to a unique horizontal curve on P_G (one whose tangents are everywhere horizontal) once a lift at one point has been specified. This is the notion of parallel transport along a curve. We shall give a more detailed description of the horizontal space in the last section of this article.

Now suppose we have the following situation: we have a reduction of P_G to an H bundle P_H and we are given a connection ω_G on P_G. Then the restriction of ω_G to P_H defines a g-valued 1-form on P_H which satisfies (1.7) with $\xi \in h$ and (1.8) with $a \in H$. As h is an invariant subspace of g, we can define the form

$$\Theta = (\text{restriction of } \omega_G \text{ to } P_H)/h$$

as a g/h-valued 1-form on P_H satisfying the conditions of vanishing on all vertical vector fields and of being equivariant with respect to the action of H.

If the group H is reductive—or more generally, if h has an H invariant complement n in g—we can decompose

$$\mathrm{g} = \mathrm{h} \oplus \mathrm{n}$$
$$\omega_G | P_H = \omega_H + \Theta$$

where we have identified n with g/h. Here ω_H is an h-valued 1-form and Θ is an n-valued 1-form. The form ω_H satisfies all the conditions for a connection on P_H. Thus:

Lemma 1.3. If the group H is reductive, then the restriction of ω_G to P_H determines an n-valued 1-form Θ on P_H and a connection ω_H on P_H. The 1-form Θ is horizontal and equivariant relative to H.

It is important to note that the horizontal subspaces for the connections ω_G and ω_H will differ, in general, at points of P_H. However, given ω_H and Θ, one can reconstruct ω_G along P_H and hence on all of P_G. Thus the data ω_H, Θ on P_H are equivalent to a connection ω_G.

In particular, if dim G/H = dim M, we can consider the condition that Θ be a soldering form. If this case holds, then ω_G is known as a *Cartan connection.* If G is the affine group, then a Cartan connection is called an *affine connection.*

Let F be a vector space on which G acts. We will let $A^k(F)$ denote the space of k-forms on M with values in $F(P)$. In particular, we can identify $A^0(F)$ with the space of sections of $F(P)$, a space which we also denote by $\Gamma(P)$. A connection ω on P defines a covariant derivative

$$\nabla^{\omega}\colon\ A^k(F) \to A^{k+1}(F)$$

$$\nabla_{\omega}\Omega = d\Omega - \omega\Omega$$

One also defines the *curvature 2-form* of the connection ω by

$$\text{curv}(\omega) = \nabla_{\omega}\omega = d\omega - 1/2[\omega, \omega] \tag{1.10}$$

For the case of a reduced bundle P_H with a reductive group H, the restriction of the curv(ω_G) to P_H is given by

$$\begin{aligned} &d\omega_H + d\Theta - 1/2[\omega_H + \Theta, \omega_H + \Theta] \\ &= d\omega_H - 1/2[\Theta, \Theta] + d\Theta - \Theta\omega_H - 1/2[\omega_H, \omega_H] \\ &= \text{curv}(\omega_H) + \nabla_{\omega}\Theta - 1/2[\Theta, \Theta] \end{aligned}$$

In particular, for the case of an affine connection, [n, n] = 0, so that the last term vanishes. The term

$$\nabla_{\omega}\Theta = d\Theta - \omega_H\Theta$$

is known as the *torsion 2-form* of the connection ω_H. In the Levi-Civita theory of connections, the torsion entered for describing the nonclosure of an infinitesimal parallelogram formed by parallel transport of infinitesimal vectors; the noncommutativity of the transport enters into the Christoffel symbols. The torsion tensor, in local coordinates, is given by the skew-symmetric components of the Christoffel symbols. But in the Cartan theory the torsion enters as a (translational) component of the curvature. In condensed matter physics, it plays the role of a dislocation density (Kleinert, 1989).

We shall assume from now on, unless otherwise stated, that all geometrical structures are infinitely differentiable, and that space-time M has dimension equal to 4. This second condition is unessential.

We shall be mainly interested in the sequel in G being the Poincaré group and H the Lorentz group, so that G/H as a homogeneous space can be identified with $R^{1,3}$, or $H = O(R^4)$ and $G = H \oplus R^4$ (in what follows, we shall not distinguish the degenerate and nondegenerate cases). Then Θ is an $R^4(P_H)$-valued soldering 1-form on M, and ω_H is an $\mathrm{h}(P_H)$-valued connection 1-form on M. Then, for any $x \in M$, $\Theta(x)$ can be thought of as a 1-form on M with values in T_xM, due to the canonical isomorphism between $R^4(P_H)$ and TM. Then, given a local coordinate system (x^α) on M, where $\alpha = 1, \ldots, 4$, we get a local coordinate system $(x^\alpha, \partial/\partial x^\alpha)$ on TM. Locally, Θ takes the form

$$\Theta(x) = (\Theta^a_\alpha(x)\, dx^\alpha) \tag{1.11}$$

with *inverse* $e^\alpha_a \partial/\partial x^\alpha$, $a = 0, 1, \ldots, 3$ representing the indices of an anholonomic basis in R^4; thus

$$\Theta^a_\alpha e^\alpha_b = \delta^a_b \qquad \text{and} \qquad \Theta^a_\alpha e^\beta_a = \delta^\beta_\alpha \tag{1.12}$$

If (g_{ab}) denotes a metric on R^4, say Euclidean or Minkowskian, the local expression for the metric that Θ puts on M is

$$g_{\alpha\beta} = g_{ab}\Theta^a_\alpha\Theta^b_\beta \tag{1.13}$$

which then has the same signature as that of (g_{ab}).

If we have a Lorentz (or orthogonal) linear connection on M, $\omega_H = (\omega^{ab}_\mu)$, then ω_H is skew-symmetric in a, b. Now, since locally any element in h takes the form $u \wedge v$, where u and v belong to V ($= R^4$ or $R^{1,3}$), then ω_H defines a connection 1-form on M with values in $\bigwedge^2 V \simeq \bigwedge^2 TM$, $\tilde{\omega}_H = (\omega^{\alpha\beta}_\mu) = (\omega^{ab}_\mu e^\alpha_a e^\beta_b)$, with associated TM-valued torsion 2-form

$$T = 1/2 T^\alpha_{\beta\gamma} dx^\beta \wedge dx^\gamma \tag{1.14}$$

with coefficients given by the torsion tensor defined by

$$T^\alpha_{\beta\gamma} = (\Theta^{-1}\nabla_\omega\Theta)^\alpha_{\beta\gamma} = e^\alpha_a(\partial_{[\gamma}\Theta^a_{\beta]} - \omega^a_{b[\gamma}\Theta^b_{\beta]}) \tag{1.15}$$

where $(\omega^a_{b\mu})$ denotes the Christoffel coefficients of the connection form ω_H induced by the isomorphism $V \otimes V = V^* \otimes V^*$ induced by g, so that $\omega^a_{b\mu} = \omega^{ac}_\mu g_{bc}$. Thus, the Christoffel coefficients of the connection 1-form $\tilde{\omega}_H$ are given by

$$\Gamma^\alpha_{\beta\mu} = e^\alpha_a\Theta^b_\beta\omega^a_{b\mu} + e^\alpha_a\partial_\mu\Theta^a_\beta \tag{1.16}$$

$\tilde{\omega}_H$ is known in Poincaré gauge theory as the *space-time linear connection.*

If we assume that ω_H is compatible with the metric on V, i.e., $\nabla_{\omega_{H\mu}} g_{ab} = 0$, then $\tilde{\omega}_H$ is compatible with the space-time metric, i.e.,

$$\nabla_{\tilde{\omega}_{H\mu}} g_{\alpha\beta} = 0 \tag{1.17}$$

Thus, lengths of vector fields are preserved under parallel transport. This means that Θ has reduced the bundle of linear frames to the orthogonal bundle.

What is essential in the connection on M defined by (1.16) is its nonsymmetric character, i.e., it has a nonzero torsion tensor

$$T^{\alpha}_{\mu\nu} = 1/2(\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}) \tag{1.18}$$

This geometry is called the Riemann–Cartan (RC) structure.

Let us introduce a conformal structure on the tangent space of M. For this we shall follow Einstein's last work (Einstein and Kaufman, 1955; Obukhov, 1982), which resolves Einstein's original criticism to Weyl's Abelian (and historically first) gauge theory of 1918.

We define the Weyl transformation on the soldering form

$$W(\Theta^{a}_{\alpha}) = \psi\Theta^{a}_{\alpha} \tag{1.19}$$

so that $W(e^{\alpha}_{a}) = (1/\psi)e^{\alpha}_{a}$, and a Weyl transformation on Γ (which by abuse of notation we denote by W, and similarly for the other derived transformations)

$$W(\Gamma^{a}_{b\mu}) = \Gamma^{a}_{b\mu} \tag{1.20}$$

Then we can *derive* the following transformation on the metric on M:

$$W(g_{\alpha\beta}) = \psi^2 g_{\alpha\beta} \qquad \text{and} \qquad W(g^{\alpha\beta}) = \psi^{-2} g^{\alpha\beta} \tag{1.21}$$

These are the well-known conformal transformations of the *metric* on M. In the above definitions, ψ is a function defined on M with values on R^+.

The Riemann–Cartan structure under the above transformations becomes

$$W(\Gamma^{\alpha}_{\beta\mu}) = \Gamma^{\alpha}_{\beta\mu} + \delta^{\alpha}_{\beta}\partial_{\mu} \ln \psi \tag{1.22}$$

with torsion tensor

$$T^{\alpha}_{\beta\mu} + 1/2(\delta^{\alpha}_{\beta}\partial_{\mu} \ln \psi - \delta^{\alpha}_{\mu}\partial_{\beta} \ln \psi) \tag{1.23}$$

This shows that only the trace of the torsion tensor is conformally transformed, i.e., the 1-form $Q = Q_{\mu}dx^{\mu} = T^{\alpha}_{\alpha\mu}dx^{\mu}$ of the original connection is transformed as $W(Q) = Q + 3/2d \ln \psi$.

There are various interesting instances of the transformation (1.22). First, if the original space-time connection coefficients are torsionless and given by the Levi-Civita coefficients determined by a space-time metric g, we have introduced torsion associated to the conformal field ψ on a purely Riemannian geometry. More general is the case in which the original torsion reduces to the trace component given by a nonexact one-form Q, which we may interpret as an electromagnetic potential with nontrivial field. Then, the above conformal transformations induce a gauge transformation so that Q is transformed to $Q + 3/2d \ln \psi$. The final case we want to consider is the one in which the original space-time connection has nonzero torsion, yet such that its trace vanishes completely. In this case, the original space-time connection is associated to the rotational degrees of freedom of the Dirac–Hestenes spinor field Ψ which produces a completely skew-symmetric torsion, while ψ is the scalar field that enters in the canonical decomposition of Ψ,

$$\Psi = \psi \exp(\gamma_5 S) R$$

where $R(x) \in \text{Spin}_+(1, 3) \cong Sl(2, C)$ with the property that $R(x)R(x)^{-1} = \text{Id}$. For the details of this we refer to Rapoport *et al.* (1994). The identification of ψ as a Schrödinger field was obtained independently from the field equations for the geometries introduced by the transformation (1.22) (Rapoport, 1994b).

Now, the connection defined by (1.22) is not metric compatible, yet the modified connection given by (we normalize the 3/2 factor)

$$\Gamma^{\alpha}_{\beta\mu} = \{^{\alpha}_{\beta\mu}\} + 2/3(\delta^{\alpha}_{\beta}\partial_{\mu} \ln \psi - g_{\beta\mu} g^{\gamma\alpha}\partial_{\gamma} \ln \psi) \tag{1.24}$$

where $\{^{\alpha}_{\beta\mu}\}$ are the coefficients of the Levi-Civita connection associated to g, is a metric-compatible connection. Then, $Q = d \ln \psi$, the logarithmic differential of the scale field ψ, is a Weyl one-form of a RC metric-compatible structure. For any metric, taking account of the fact proved in Rapoport (1991) that ψ^2 is an invariant probability density, then $2K_B Q$, where K_B is Boltzmann's constant, can be thought of as an entropy one-form. We note here that consequently ψ need not be a smooth function and then Q is nonsmooth, following the already classical theory of distributional differential forms due to de Rham (de Rham, 1984).

The metric compatibility of these RC structures produced by the general action of the conformal group distinguishes them from the usual Weyl geometry produced by the transformations on the space-time metric (1.21). In the latter, it is the Weyl one-form which precisely expresses the lack of preservation of lengths under parallel Weyl transport. So the introduction of these structures solves a long-standing problem of compatibility of the RC structures with the local action of the Weyl group.

Therefore, this geometry, which we shall call Riemann–Cartan–Weyl (RCW), does not have the *historicity problem* which moved Einstein to reject Weyl's attempt to construct the first gauge theory in which he associated the Weyl form to the electromagnetic field, this in spite of Q not being a complex field (yet, a nonobvious association).

It is of great importance that our above constructions can be carried out for the case of a general configuration space M of dimension m, on taking instead of the Poincaré group the group given by the semi-direct sum $O(m) + R^m$. This is of relevance for the formulation of quantum mechanics for a system of n particles for which M is their configuration manifold (so that $m = 4n$ in the relativistic case) and ψ denotes their *joint* wave function, and is valid also for the Dirac–Hestenes spinor fields (Rapoport *et al.*, 1994).

2. THE D'ALEMBERT AND WAVE OPERATORS OF THE RCW STRUCTURES

In this section we construct the Laplacian operators associated to RCW geometries. As explained in the Introduction, the relevance of this follows from the fact that these operators play a central role in the formulation of quantum mechanics as a theory of Brownian motion, as well as a Hilbert space operator theory.

We shall study, then, the D'Alembert—in the case g is Lorentzian—or Laplacian—in the Riemannian case—operator associated to the RC structures (of course, for diffusion processes, we only consider the Riemannian case). The construction we shall present is valid for both cases, and from now on we shall simply speak of Laplacian operators. Our treatment differs from the original treatment given in Rapoport (1991) and follows Rapoport (1995b, c). A completely different derivation of this operator can be found in Kleinert (1991).

Henceforth, in this section the dimension of M will be an arbitrary n. We start with an RC connection described by an arbitrary metric g and an arbitrary torsion tensor. Let ∇ denote its covariant derivative operator, which we additionally assume to be compatible with g, i.e., $\nabla g = 0$. Denote the Christoffel coefficients of ∇ as $\Gamma^{\alpha}_{\beta\gamma}$; then,

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} + 1/2K^{\alpha}_{\beta\gamma} \tag{2.1}$$

where the first term in (2.1) stands for the Christoffel Levi-Civita coefficients of the metric g, and

$$K^{\alpha}_{\beta\gamma} = T^{\alpha}_{\beta\gamma} + S^{\alpha}_{\beta\gamma} + S^{\alpha}_{\gamma\beta}$$

is the cotorsion tensor, with

$$S^{\alpha}_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^{\kappa}_{\nu\gamma}$$

Let us consider the Laplacian operator on functions associated to this Cartan connection, defined—extending the usual definition—by

$$H(\nabla)f := 1/2 \operatorname{tr}(\nabla^2)f = 1/2 g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta} f \tag{2.2}$$

for any continuously differentiable function f on M, where ∇ stands for the covariant derivative operator with respect to Γ. A straightforward computation shows that $H(\nabla)$ only depends in the trace of the torsion tensor and g:

$$H(\nabla) = 1/2\Delta_g + g^{\alpha\beta}Q_\beta\partial_\alpha \tag{2.3}$$

with $Q = T^{\nu}_{\nu\beta}dx^\beta$, the trace-torsion one-form, and

$$\Delta_g = \mathrm{tr}((\nabla^g)^2) = (\det g)^{-1/2}g^{\alpha\beta}\partial_\alpha((\det g)^{1/2}g^{\alpha\beta}\partial_\beta)$$

the Laplace–Beltrami operator associated to the Levi-Civita connection ∇^g. Therefore, for the Riemann–Cartan connection ∇ we have that, on smooth functions defined on M,

$$H(\nabla) = \frac{1}{2}\,\mathrm{tr}((\nabla)^2) = \frac{1}{2}\,\Delta_g + \hat{Q} \tag{2.3$'$}$$

with $\hat{Q}$ the vector-field dual to the 1-form Q: $\hat{Q}(f) = \langle Q, \mathrm{grad}\, f\rangle, f: M \to R$.

Notice that $H(\nabla)$ only depends on g and the trace-torsion of the connection ∇; the other terms of the invariant decomposition of the torsion tensor of ∇ do not appear in the Laplacian. Thus, to obtain a one-to-one correspondence (in general dimension other than 2, when this is satisfied trivially) between Cartan connections and their Laplacian operators, we restrict ourselves to ∇ with Christoffel symbols of the form

$$\Gamma^\alpha_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + \frac{2}{(n-1)}\,\{\delta^\alpha_\beta Q_\gamma - g_{\beta\gamma}Q^\alpha\} \tag{2.4}$$

We have

$$H(\nabla) = \frac{1}{2}\,\mathrm{tr}((\nabla^g)^2) + \hat{Q} \tag{2.5}$$

In the case that g is Riemannian, then the expression (2.3) is the most general invariant Laplacian acting on functions defined on a smooth manifold associated to a Markovian semigroup that preserves probability (Rapoport, 1995c). This restriction will allow us to establish a one-to-one correspondence between Riemann–Cartan–Weyl connections (2.4) with Markovian diffusion processes.

We shall further assume in the following that Q reduces to the exact form $Q = d \ln \psi$, where ψ is a real function on M. In this case, the RCW geometry is determined by the Riemannian metric g and the function ψ. The corresponding Laplacian, which we shall write from now on as $H(g, \psi)$, is defined by its action on the function $f: M \to R_{>0}$ by

$$H(g, \psi)f = \frac{1}{2}\,\Delta_g f + \langle \mathrm{grad} \ln \psi, \mathrm{grad}\, f\rangle \tag{2.6}$$

The theory we shall construct is determined by this Laplacian, which we shall call the RCW Laplacian. The more general case of non-exact Q is related to irreversibility of the quantum motions. It will be discussed in detail in a forthcoming article; see also (Rapoport, 1995d).

3. RCW LAPLACIANS AND WITTEN'S DEFORMED LAPLACIAN OF TOPOLOGICAL QUANTUM FIELD THEORIES

Let us assume in the following that we have a smooth n-dimensional orientable compact manifold M provided with a Riemannian metric g. We consider the Hilbert space of square-summable differential forms of degree q on M with respect to vol_g. We shall denote this space as $L^{2,q}$ or as $L^2\Omega^q(M, \mathrm{vol}_g)$. The inner product in $L^{2,q}$ is

$$\langle \omega, \phi \rangle = \int_M \langle \omega(x), \phi(x) \rangle \, \mathrm{vol}_g$$

where the integrand is given by the natural pairing between the components of ω and the conjugate tensor

$$g^{\alpha_1\beta_1} \cdots g^{\alpha_q\beta_q}\phi_{\beta_1\cdots\beta_q}$$

Alternatively, we can write in a coordinate-independent way $\langle \omega(x), \phi(x) \rangle \, \mathrm{vol}_g = \omega(x) \wedge * \phi(x)$, with $*$ the Hodge star operator, for any $\omega, \phi \in L^{2,q}$.

The de Rham–Kadaira operator on $L^{2,q}$ is defined as

$$\Delta = -(d + \delta)^2 = -(d\delta + \delta d)$$

where δ is the formal adjoint defined on $L^{2,q+1}$ of the exterior differential operator d defined on $L^{2,q}$:

$$\langle \delta\phi, \omega \rangle = \langle \phi, d\omega \rangle$$

for $\phi \in L^{2,q+1}$ and $\omega \in L^{2,q}$. In the case of $q = 0$, this is the Laplace–Beltrami operator on functions encountered before; in the general case we have in addition to $\mathrm{tr}(\nabla^g)^2$ the contribution of the Weitzenböck curvature term. Let us be given a C^2 positive function ψ on M. We then have an induced smooth density $\rho = \psi^2 \, \mathrm{vol}_g$ on M.

We introduce the Hilbert space $L^{2,q,\rho} = L^2\Omega^q(M, \rho)$ of differential forms on M of degree q, square integrable with respect to ρ, with inner product

$$\langle \phi_1, \phi_2 \rangle_\rho = \int_M \langle \phi_1(x), \phi_2(x) \rangle \rho \tag{3.1}$$

for $\phi_1, \phi_2 \in L^{2,q,\rho}$. We define the quadratic form $q(\phi) = \frac{1}{2}\langle \phi, \phi \rangle_\rho$, with ϕ

on the Hilbert space given by the completion of the space of all smooth q-forms under the $L^{2,\rho}$ inner product. In the case of exact one-forms, this is the quadratic form introduced in correspondence with the Brownian processes determined by $H(g, \psi)$ (Rapoport, 1991, 1995a, b, c).

Consider the formal adjoint of d, which we shall denote as δ^{ψ}, defined on $L^{2,q+1,\rho}$ as follows:

$$\langle \delta^{\psi}\omega, \phi \rangle_{\rho} = \langle \omega, d\phi \rangle_{\rho} \tag{3.2}$$

for any $\omega \in L^{2,q,\rho}$ and $\phi \in L^{2,q,\rho}$. Since $d^2 = 0$, we have

$$(\delta^{\psi})^2 = 0 \tag{3.3}$$

For any smooth function f defined on M, and ω a q-form

$$\delta(f\omega) = f\delta\omega - i_{\text{grad} f}\omega$$

where i_X is the interior product derivation on q-forms.

We introduce the operator on $L^{2,q,\rho}$:

$$\Delta^{\psi,q} = -(d + \delta^{\psi})^2 \tag{3.4}$$

which we can write as

$$-(d\delta^{\psi} + \delta^{\psi} d)$$

Recalling the definition of the Lie-derivative operator $L_X = di_X + i_X d$, X a smooth vector field on M, we finally have

$$\Delta^{\psi,q} = \Delta^q + 2L_{\text{grad} \ln \psi} \tag{3.5}$$

Here Δ^q denotes the de Rham–Kodaira leplacian on q-forms.

Let us define now the deformed exterior differential operator mapping q-forms into $q + 1$-forms, by

$$d^{\psi} = \psi d \psi^{-1} \tag{3.6}$$

so that

$$d^{\psi}\omega = d\omega - d \ln \psi \wedge \omega$$

We have that

$$(d^{\psi})^2 = 0 \tag{3.7}$$

This operator is the ($\tau = -1$ version) of Witten's deformed differential (Witten, 1982). We introduce now the deformed co-differential operator as the formal adjoint of d^{ψ}:

$$(d^{\psi})^* = \psi^{-1}\delta\psi \tag{3.8}$$

We introduce the Witten deformed Laplacian operator, defined as

$$L^{\psi,q} = -(d^{\psi} + d^{\psi *})^2 \tag{3.9}$$

which can still be written as

$$-(d^{\psi}d^{\psi *} + d^{\psi *}d)$$

We have the following relation between the two Laplacian operators:

$$\Delta^{\psi,q} = \psi^{-1}L^{\psi,q}\psi \tag{3.10}$$

so that these two operators are conformally equivalent under conjugation by ψ. Note that $\Delta^{\psi,0} = 2H(g, \psi)$.

The key to the construction of quantum mechanics as a theory of diffusion processes and a Hilbert space operator theory rests on choosing the operators $\frac{1}{2}\Delta^{\psi,q}$ as infinitesimal generators of Markovian semigroups,[2] P^q_τ, $q = 0, \ldots, n$. Here P^0_τ is the stochastic process with infinitesimal generator given by $H(g, \psi)$, which was constructed in Rapoport (1991, 1995a, b). The construction of this family of diffusion processes on differential forms of arbitrary degree rests on the knowledge of the data g and ψ which determine the RCW structure; these data are determined from a stochastic extension of the Einstein–Hilbert variational principle to RC geometries; one proves that ψ is a solution of the conformal invariant wave equation (Rapoport, 1995a).

Thus, starting from the RCW geometry determined by the field equations, we can construct a family of stochastic processes on forms of any degree. Remarkably, the Laplacian introduced by Witten in topological quantum field theory appears to be related to a wave function which satisfies the field equations and produces the torsion of the RCW geometry.

We would like to note finally that from the fact that $(d^{\psi})^2 = 0$ we can define a deformed de Rham complex $H^q_{\psi}(M, R)$ as

$$\mathrm{Ker}(d^{\psi}: \Lambda^q \to \Lambda^{q+1})/\mathrm{Ran}(d^{\psi}: \Lambda^{q-1} \to \Lambda^q)$$

Since $\mathrm{Ker}(d^{\psi}) = \psi\ \mathrm{Ker}(d)$ and $\mathrm{Ran}(d^{\psi}) = \psi\ \mathrm{Ran}(d)$, we obtain that $H^q_{\psi}(M, R) \cong H^q(M, R)$ for any $q = 0, \ldots, n$. Now, by Hodge's theorem, $\dim H^q(M, R) = \dim(\mathrm{Ker}(\Delta^q))$, which by the above construction is clearly equal to $\dim(\mathrm{Ker}(L^{\psi,q}))$; by (3.10) we conclude that

$$\dim(H^q(M, R)) = \dim(\mathrm{Ker}\ \Delta^{\psi,q}) \tag{3.11}$$

This identification, which we shall not use in this article, is fundamental to

[2] A Markovian semigroup in a Hilbert space H is a family of bounded positive linear operators $\{P_\tau, \tau \geq 0\}$ with dense domain contained in H such that $P_0 = \mathrm{Id}$ satisfying the following properties: (i) (semigroup property) $P_\tau \circ P_{\tau'} = P_{\tau+\tau'}$, $\tau, \tau' \geq 0$, (ii) (contraction property) $\|P_\tau\| \leq 1$, $\tau \geq 0$, an (iii) $\tau \to P_\tau$ is strongly continuous.

the formulation of the ergodic studies of the flows; indeed, if the first Betti number of M is $b_1(M) = \dim(H^1(M, R)) \neq 0$, then it can be proved that the flows corresponding to RCW geometries with infinitesimal generators given by $\frac{1}{2}H(g, \psi)$ are (moment) unstable (Rapoport, 1995b,c).

5. RCW GEOMETRIES AND SUPERSYMMETRIC SYSTEMS

That Laplacian operators on smooth compact manifolds are examples of supersymmetric systems was a profound observation due to Witten (1982).

One starts with a Hamiltonean H on a Hilbert space **H**, together with a self-adjoint operator Q and a bounded self-adjoint operator P both defined on **H**, such that

$$H = Q^2 \geq 0, \qquad P^2 = 1, \qquad \{Q, P\} = QP + PQ = 0$$

Then the triple $\{H, P, Q\}$ is said to be a supersymmetric system, or to have supersymmetry. Since P is self-adjoint and $P^2 = 1$, then P has 1 and -1 for eigenvalues. Denote

$$\mathbf{H}_{\text{ferm}} = \{\phi \in \mathbf{H}, P\phi = -\phi\}$$

and

$$\mathbf{H}_{\text{bos}} = \{\phi \in \mathbf{H}, P\phi = \phi\}$$

which are called the fermionic and bosonic states, respectively. Then, Q: $\mathbf{H}_{\text{ferm}} \to \mathbf{H}_{\text{bos}}$ and Q: $\mathbf{H}_{\text{bos}} \to \mathbf{H}_{\text{ferm}}$, or in other words, Q maps fermionic states into bosonic states and conversely.

In the present theory, we take for Hilbert space $\mathbf{H} = \oplus_{q=0}^{n} L^{2,q,\rho}$, and the Hamiltonean operator H is $\Delta^{\psi} = \frac{1}{2}\Delta + L_{\text{grad} \ln \psi}$ as an operator on forms of arbitrary degree, where $\Delta = -(d\delta + \delta d)^2$. Now we take $Q = i(d + \delta^{\psi})$ and P is defined on **H** by its restriction to q-forms: $P \mid L^{2,q,\rho} = (-1)^q$, $q = 0, \ldots,$ n, i.e., the operator of multiplication by $(-1)^q$. Then it is easily seen that $\{H, P, Q\}$ is a supersymmetric system. Thus, in this setting, fermionic (bosonic) states are given by odd (even) forms.

We remark that the quantization of gravitation suggested in the above sections by taking $\frac{1}{2}\Delta^{q,\psi}$, with $q = 0, \ldots, n$, for infinitesimal generators of diffusion processes—corresponding to fermions and bosons—determined by solving the heat kernel of each Markovian family P^q, $0 \leq q \leq n$, depends on the knowledge of g and ψ. Thus, the knowledge of the RCW geometry determines the quantum theory for bosons and fermions alike.

6. RCW GEOMETRIES AND THE SYMPLECTIC STRUCTURE OF LOOP SPACE

It may seem rather strange that in the present theory, in contrast with TQFT, ψ is defined on space-time M instead of being a functional on loop space, i.e., the infinite-dimensional manifold $\Omega = \{\phi: S^1 \to M, \phi \in C^\infty\}$. In this description, M can be recovered as the constant loops of Ω. This infinite-dimensional setting is the one considered in topological quantum field theories, and different choices of ψ yield supersymmetric σ models, supersymmetric ϕ^4 theory, etc. (Witten, 1982). Yet, it is to be remarked that the RCW geometries are connected to the symplectic structure on Ω, in a way we describe in the following.

Let us fix $\phi \in \Omega$. Then, the tangent space to Ω at ϕ, $T_\phi\Omega$, can be identified with the space of sections of the pullback vector bundle $\phi^*(TM)$ of TM to S^1 by ϕ. The metric g on M defines a metric $\cdot_\phi$ on $\phi^*(TM)$, and hence we have an inner product on $T_\phi\Omega$: $(s_1, s_2) = (1/2\pi) \int_{S^1} s_1 \cdot_\phi s_2$. Thus, $T_\phi\Omega$ has a pre-Hilbert structure.

Next, we introduce a general Riemann–Cartan connection ∇ on M. This induces a connection on $\phi^*(TM)$, and hence a covariant derivative operator ∇_ϕ which acts on sections of $\phi^*(TM)$ by evaluation on the vector field $d/d\sigma$ of S^1. Now we can define a skew-symmetric bilinear form on $T_\phi\Omega$:

$$\omega(\phi) = \frac{1}{4\pi} \int_{S^1} (\nabla_\phi s_1 \cdot_\phi s_2 - \nabla_\phi s_2 \cdot_\phi s_1) \tag{6.1}$$

Varying $\phi \in \Omega$, we obtain a differential 2-form on Ω. As first noted by Atiyah (1985), $d\omega$ equals $(1/2\pi)$ times the integral over S^1 of the skew-symmetric component of the torsion tensor; for details, see Bowick and Rajeev (1987).

Therefore, for an RCW geometry, ω is a closed 2-form on Ω. Yet, ω is not properly a symplectic form, since it vanishes at those ϕ for which ∇_ϕ has a 0 eigenvalue, i.e., on any tangent vector which is covariantly constant along ϕ. In the case of a purely Riemannian geometry, i.e., $\psi = 1$, this "symplectic" setting has been applied to obtain an exact computation of the trace of the heat kernel of $2\pi\Delta_g$ and to directly obtaining the Atiyah–Singer index theorem (Atiyah, 1985). It would be interesting to check if this construction can be carried out for the trace of the heat kernel of $H(g, \psi)$.

We close this section with the observation that the 2-form of (6.1) is related to the zero modes of string theory (Bowick and Rajeev, 1987); the relevance of RC geometries to string theory was assessed in Scherk and Schwarz (1974).

7. THE CARTAN CLASSICAL COPYING METHOD

In closing this first in a series of articles, we wish to describe the Cartan classical copying method as a preparation for its stochastic extension (Rapoport, 1995d).

For this, we need to study in further detail the structure of horizontal vector fields on the bundle of orthogonal frames P_H, which we shall do next.

The total space of P_H is described as the space of all pairs $r = (x, e)$, where $x \in M$ and $e = [e_1, \ldots, e_n]$ is an orthonormal frame at x, i.e., the vectors $e_1, \ldots, e_n$ are a basis for T_xM satisfying the condition

$$g_{\alpha\beta}e^\alpha_a e^\beta_b = \eta_{ab} \tag{7.1}$$

where $\eta = (\eta_{ab})$ is the Minkowski metric in the case g is Lorentzian (or the Euclidean metric in the Riemannian case); as we already saw,

$$e^\alpha_a e^\beta_b = g^{\alpha\beta} \tag{7.2}$$

Every vector field L on M induces a vector field $\tilde{L}$ on P_H, defined as follows. If f is a smooth function on P_H, then $\tilde{L}f$ is given by

$$(\tilde{L}f)(r) = \frac{d}{dt} f((\exp tL)x, (\exp tL)_* e)\big|_t = 0$$

where $r = (x, e)$ and $(\exp tL)_*$: $T_xM \to T_{(\exp rL)x}M$ is the tangent mapping to $\exp tL$, so that

$$(\exp tL)_*(e) = [(\exp tL)_* e_1, \ldots, (\exp tL)_* e_n]$$

Here, we recall, $\exp tL$ is the local diffeomorphism $x \to x(t, x)$ of M defined by the flow of the differential equation

$$\frac{dx^\alpha}{dt} = a^\alpha(x, t), \qquad \text{where} \quad L = a^\alpha(x)\partial_\alpha$$

$$x(0, x) = x$$

If in P_H we have a Cartan connection ∇ with coefficients $(\Gamma^\alpha_{\beta\gamma})$ which is compatible with g, we can describe the horizontal subbundle of P_H at each $r \in P_H$ as

$$H_r = \{X \text{ is a vector field on } P_H\text{: } X = A^\alpha(x)\partial_\alpha - \Gamma^\alpha_{\beta\gamma}(x)e^\gamma_a A^\beta(x)\partial/\partial e^\alpha_a\}$$

H_r is clearly a subspace of $T_r(P_H)$, which is clearly independent of the choice of local coordinates (x^α, e^α_a) on P_H. We recall that the horizontal lift $\tilde{\xi}$ of a vector $\xi \in T_xM$ is uniquely determined by $\pi_*(\tilde{\xi}) = \xi$ and $\pi(e) = x$.

Given a vector field X on M, the horizontal lift $\tilde{X}$ is the unique vector field of such that $\tilde{X}_r$ is the horizontal lift of $X(\pi(r))$ for all $r \in P_H$. In a local coordinate system (x^α, e^α_a), if $X = X^\alpha(x)\partial_\alpha$, then

$$\tilde{X} = X^\alpha \partial_\alpha - \Gamma^\gamma_{\alpha\beta} X^\alpha e^\beta_a \frac{\partial}{\partial e^\gamma_a} \tag{7.3}$$

Let Θ: $[0, \infty) \to M$ be a smooth curve on M; then the horizontal lift $\tilde{\Theta}$ of Θ is the unique curve on P_H such that $d\tilde{\Theta}/dt(t)$ is horizontal and for any $t \geq 0$, we have that $\pi(\tilde{\Theta}(t)) = \Theta(t)$. Clearly, if $r = (x, e)$ is given, where $x = \Theta(0)$, then a horizontal lift $\tilde{\Theta}$ of Θ starting at r exists and is unique. Indeed $\tilde{\Theta}(t) = (\Theta(t), [e_1(t), \ldots, e_n(t)])$, where $e_a(t) \in T_{\Theta(t)}M$ is obtained from e_a by parallel transport along Θ through $\tilde{\nabla}$, for each $a = 1, \ldots, n$; there exists a unique vector field $\tilde{L}_a$ on P_H such that $(\tilde{L}_a)_r$ is the horizontal lift of $e_a \in T_xM$ for every $r = (x, e)$ and $a = 1, \ldots, n$. In local coordinates as above,

$$\tilde{L}_a = e^\alpha_a \partial_\alpha - \Gamma^\delta_{\beta\gamma} e^\beta_a e^\gamma_b \frac{\partial}{\partial e^\delta_b} \tag{7.4}$$

The set $\{\tilde{L}_1, \ldots, \tilde{L}_n\}$ is called the system of horizontal vector fields, or basic vector fields.

We are finally ready to introduce the Cartan classical copying method. Let M be a manifold provided with a Cartan connection ∇ compatible with a metric g on M. The Cartan connection enables us to roll M along a curve $\gamma(t)$ on R^n (where n is the dimension of M) to obtain a curve $\Theta(t)$ on M as the trace of the curve γ. An important comment is in order: γ is arbitrary, i.e., it can be the trajectory of an arbitrary dynamical system. The method we shall present works because the tangent space at every point is identical to the quotient of the affine group of H (H direct sum with translations) with H, so that we can place a Cartan connection ∇ on P_H. To make this precise, let $r = (x, e) \in P_H$ and let γ: $[0, \infty) \to R^n$ be a smooth curve. Define $\tilde{\Theta}$: $[0, \infty) \to P_H$, where $\tilde{\Theta}(t) = (\Theta(t), e(t))$ by

$$\text{(copying equation)} \qquad \frac{d\Theta}{dt}(t) = e_a(t) \frac{d\gamma^a}{dt}(t) \tag{7.5}$$

$$\text{(parallel transport)} \qquad \tilde{\nabla}_{d\Theta/dt} e_a(t) = 0 \tag{7.6}$$

for every $a = 1, \ldots, n$, with the initial conditions

$$\Theta(0) = x, \qquad e(0) = e \tag{7.7}$$

Equations (7.5) and (7.6) have the componentwise forms

$$\frac{d\Theta^\alpha}{dt}(t) = e^\alpha_a(t) \frac{d\gamma^a}{dt}(t) \tag{7.6$'$}$$

$$\frac{de^\alpha_a}{dt}(t) = -\Gamma^\alpha_{\beta\gamma}(\Theta(t)) e^\gamma_a(t) \frac{d\Theta^\beta}{dt}(t) \tag{7.7$'$}$$

which are the componentwise expressions of the single equation

$$\frac{d\tilde{\Theta}}{dt}(t) = \tilde{L}_a(\tilde{\Theta}(t))\,\frac{d\gamma}{dt}(t) \tag{7.8}$$

and the initial condition for $\tilde{\Theta}$ is

$$\tilde{\Theta}(0) = r \tag{7.9}$$

where $\tilde{L}_a$ is the system of horizontal vector fields on P_H. The solution curve $\Theta(t) = \pi(\tilde{\Theta}(t))$ depends on the initial frame at x; we shall denote it by $\Theta(t) = \Theta(t, r, \gamma)$, with $r = (x, e)$. It follows easily that $\Theta(t, Ar, \gamma) = \Theta(t, r, A\gamma)$ for any $t \in [0, \infty)$ and $A \in H$, where the curve $A\gamma$ is defined by $(A\gamma)(t) = A\gamma(t)$.

Thus we have completed the formulation of the Cartan copying method for classical curves, i.e., for smooth curves. We shall call this instance of the Cartan method the *classical Cartan copying method.*

The striking point is that one can generalize this method to the copying of Brownian motions in R^n. It is more striking still that to construct quantum mechanics as a theory of diffusions, one does not start with an arbitrary continuous stochastic process on R^n, but the simplest elementary case of a homogeneous isotropic process, the Wiener process, whose transition probability is the standard Gaussian density on R^n. The diffusion process on M becomes completely determined by the RCW connection given by only the trace part of the torsion of ∇. To carry over this construction to the stochastic case, we are lacking one point: the substitution of the usual rules of calculus (more specifically, a chain rule) for smooth functions taken along smooth curves for rules that are applicable in the case that the curves are sample paths of Wiener processes which are continuous nondifferentiable (furthermore, they are fractals). This shall be presented in a forthcoming article, together with the formulation of quantum mechanics as Dirichlet forms associated to the RCW Laplacians.

8. CONCLUSIONS

We have constructed the RCW geometries and their associated Laplacian operators in view to the construction of quantum mechanics as diffusion processes or still, as Markovian semigroups having these Laplacians as infinitesimal generators. In this article we have not presented the field equations for the RCW geometries, which are essentially related to the solution of a Dirichlet problem for the conformal invariant wave equation in the canonical Hilbert space determined by the volume form. Thus, the quantization of gravitation envisaged in this program appears to be related to the usual quantization scheme through the heat kernel expansion through *Riemannian invariants* (Fulling, 1989; Birell and Davies, 1982).

What matters centrally in this quantization, and has been unnoticed by other authors, is that from the field equations for the RCW geometry, the quantum potential is found to be no other than 1/12 the metric scalar curvature, which has assimilated the dependence of the quantum system on the square root of the invariant density ψ^2. In fact, one proves that the explicit dependence of the quantum system on the RCW geometry shows up while working in the Hilbert space $L^2(\psi^2 \text{ vol}_g)$, where vol_g is the canonical volume density associated to the metric g, and $\psi^2 \text{ vol}_g$ is the invariant density of the quantum diffusion. By conformal transformation to the canonical Hilbert space $L^2(\text{vol}_g)$, the heat kernel representation of the diffusion in the *RCW* Hilbert space goes to the heat kernel representation for the conformal invariant wave operator on the canonical Hilbert space (Rapoport, 1995c). Thus, in the $L^2(\psi^2\text{vol}_g)$ Hilbert space the role of torsion is essential, while in the Hilbert space $L^2(\text{vol}_g)$ the role of torsion is lost due to the identity of the quantum potential with $(1/12)R(g)$, where $R(g)$ is the metric scalar curvature. This settles the question as to the Riemannian or Cartanian character of the geometry of quantum mechanics and gravitation. It seems that Anandan (1988) was the first author to point out that the London description of quantum mechanics is related to the Weyl geometry, yet that this does not rule out a possible need of incorporating torsion into quantum mechanics.

As a closing remark, we point out that the present theory leads to the formulation of the ergodicity studies of the diffusion processes generated by the RCW geometries. The striking fact that allows for such a formulation is that the flows of the diffusion processes generated by the RCW geometries are *diffeomorphisms* of space-time, in spite of the fact that they arise from a nondifferentiable dynamics. As a result of this, the evolution of densities governed by a quantum Perron–Frobenius semigroup leads to the fact that the tensor product of the Wiener measure with the Born invariant measure of the diffusions, $\psi^2\text{vol}_g$, yields an equilibrium measure for the RCW diffusions (Rapoport, 1995b, c).

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